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Vertex corrections in the theory of foam drainage

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Abstract

The experimental results of Koehler and co-workers have indicated that vertices (nodes) can in some cases play a dominant role in the dissipative process that controls foam drainage. We present calculations of numerical constants that can express the effect of the vertices, in a first approximation. Two limiting cases are treated: Poiseuille flow, and free-boundary flow (in the sense that the stress at the surface of the Plateau borders is everywhere zero). Consequences for the relationship between average flow velocity and volume flow rate in steady drainage are indicated.

1. Introduction

Foam drainage is the term used for the transport of liquid through a foam, driven by gravity or pressure differences and resisted by dissipative forces. It is of direct practical importance to industry. At a more basic level it raises interesting questions regarding the role and properties of the surfaces that confine the liquid within the foam. To what extent are these liquid–gas interfaces mobile? May they be considered as fixed boundaries for liquid transport (the Poiseuille condition), expanding and contracting in response to pressure changes but not moving laterally in response to shear stress in the liquid?

Most theories have adopted the model of Poiseuille flow, with little direct justification. In such a model the Plateau borders constitute a network of channels for flow, and films play no part in the liquid transport.

For relatively dry foams these channels meet in symmetric fourfold intersections (see figure 1). This allows a theory to be developed in a straightforward way, leading to a foam drainage equation which has interesting analytic and numerical solutions [2]. Evidence for the validity of this model supports it at least semi-quantitatively in many cases, as described below [3], and it has been applied extensively [4–7].

In particular, the forced drainage experiment imposes a steady downward flow under gravity. The liquid fraction Φ_l is also uniform, to within a good approximation, as found in the much earlier work of Lemlich and co-workers [8] on steady drainage. The other quantities of interest are the average velocity of flow v, and the volume flow rate Q. By definition these are subject to

$$Q = A_t v \Phi_l \tag{1}$$

where A_t is the cross-sectional area of the vessel containing the foam.

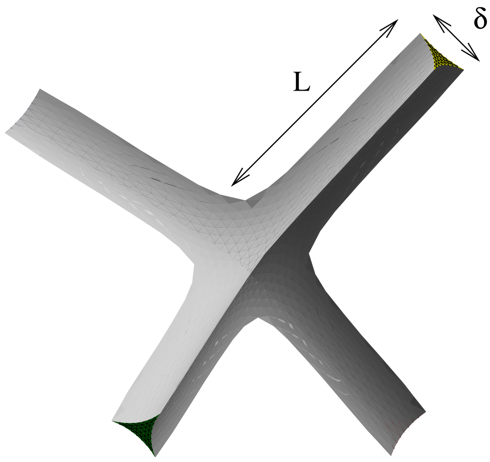

Figure 1. A single tetrahedral junction of Plateau borders. The radius of curvature of the Plateau borders is equal to δ (as length $L \to \infty$).

In practice v is often measured as the velocity of the front of the solitary wave that is generated when a given flow is first imposed on the dry foam. The earliest experiments of this kind [9] established a scaling law between v and Q:

$$v \propto Q^{1/2} \tag{2}$$

where the exponent was established to within about ten per cent. This result, later confirmed for various detergent systems [10, 11] and protein foams [12, 13], was the spur to subsequent theoretical analysis. The foam drainage equation based on Poiseuille flow was found to be consistent with (2).

However, fresh experimental results obtained in 1999 [1] lead to a reappraisal of the model. The new data, which were of greater extent and precision than previous results, indicated a different power law:

$$v \propto Q^{1/3}. \tag{3}$$

Koehler *et al* showed this to be consistent with an alternative model in which dissipation is dominated by the vertices or nodes where the Plateau borders meet. The implication is that there is plug flow rather than Poiseuille flow in the borders themselves.

While the earlier conclusions were based on data of a lesser accuracy, they had been independently confirmed many times. There was therefore a sharp conflict of evidence, which was soon resolved by the realization that different surfactants were used by the two groups [5]. The new experiments used the commercial detergent Dawn whereas the earlier work used (mostly) Fairy Liquid. Not all dishwashing detergents are the same!

On closer examination, most of the experimental results deviate somewhat from the ideal values 1/3 and 1/2 for the index which is at issue. They mostly lie between these extremes. This calls for a combined model [12, 14], and indeed further experimentation on a wide range of surfactant systems which are better defined.

Now that the vertices are seen as important, it is useful to calculate the flow properties associated with them. Here we present numerical calculations for the two limiting cases—Poiseuille flow and free-boundary flow. Calculations for the Poiseuille case have been previously reported by Pertsov *et al* [15], but not in a form readily amenable to comparison with ours.

The contribution of the vertices to the liquid fraction may also be calculated. Boltenhagen and Pittet [16] have suggested the importance of vertex contributions to the liquid fraction in drainage experiments. We would insist that this cannot be treated in isolation, since the effect of vertex corrections to fluid flow is of the same order.

The general approach adopted here is that of Phelan *et al* [17] who addressed the calculation of vertex corrections for electrical conductivity, which is closely analogous [18]. Whereas a boundary integral method was employed in that case, here we use the computational fluid dynamics package *Fluent*. This takes a meshed volume and applies a control-volume-based technique to solve the equations of fluid motion for the given boundary conditions. After the iteration process is complete, quantities such as the mass and volume flow rates may be calculated.

As in the version of the analysis of electrical conductivity due to Phelan *et al* [17], we shall express the effect of the contribution of the vertex in Poiseuille flow by incorporating it in the adjoining Plateau borders, as described below. The relevant constants are therefore effective lengths or length corrections for Plateau borders. In the case of free-boundary flow, we shall calculate a resistance parameter, which is independent of the length of the adjoining Plateau borders.

The constants will enable us to comment further on the comparison of theory and experiment. Of itself, this adds no insight to the question of the factors which dictate the behaviour of the surfaces, which remain puzzling. Surface viscosity has often been invoked as the key property determining the boundary conditions [19, 20], but this does not seem to have been demonstrated. The true picture may be much more complicated.

2. Formalism

While it is obvious that the liquid fraction may be corrected for the vertex contribution by attributing a slightly increased effective length to uniform Plateau borders, it is not so clear for electrical or flow resistance. Accordingly the rationale will be sketched.

2.1. Poiseuille case

We are concerned with a fairly dry foam, in which narrow Plateau borders, of length L (not always the same for all Plateau borders in the foam) and of width δ, meet in junctions which are also of extent δ. The assumption $\delta \ll L$ underlies much of what is assumed below, without always being explicitly stated. In the limit of a dry foam the borders may be represented as lines, meeting at intersections where there is perfect tetrahedral symmetry. Unit line length is then associated with a certain volume of liquid (of order δ^2), a certain electrical conductance (of order δ^2) and a certain flow conductance (of order δ^4), the ratio of volume flow to pressure difference [18]. A body force due to gravity is equivalent to a pressure difference, in the usual way. Of course, in the case of free boundaries this flow conductance is infinite.

The simple description fails at or close to the vertex. This may simply be ignored for Poiseuille flow, on the grounds that the effect is negligible, being of higher order in the small quantity δ/L. Indeed, that follows explicitly from the present analysis.

Recall the linear relationship between flow rate Q and pressure difference Δp over a length L of a single Plateau border:

$$Q = R_{pb}^{-1} \frac{\Delta p}{L} \qquad \text{where } R_{pb} = \frac{\chi \eta}{A_{pb}^2} \tag{4}$$

is the flow resistance per unit length of a single Plateau border in Poiseuille flow. Here the cross-sectional area of the border is denoted by A_{pb}, $\chi \approx 49$ is a further geometrical factor [8] and η is the liquid viscosity.

Now that we wish to retain vertex effects, at least in a first approximation, we account for them as follows. Each vertex is represented in the calculation of liquid volume, electrical resistance and flow resistance by the addition of a corresponding length correction to each adjoining border. This is indicated schematically in figure 2. In the case of either flow or electrical resistance, this length correction is found to shorten the Plateau borders. The proof of the validity of this procedure rests on symmetry—we will couch it in terms appropriate to the case of liquid flow.

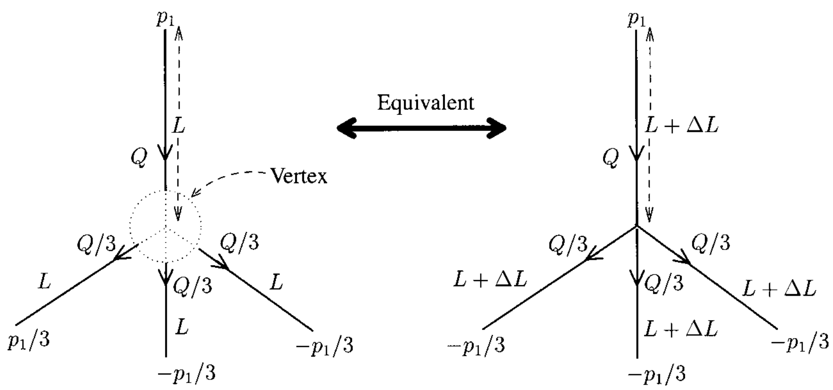

Figure 2. A schematic representation of the method of length correction. Note that for a volume correction ΔL is positive, while for a resistance calculation it is negative.

The flow properties of a vertex may be expressed by the relation (assumed linear) between four flow rates q_i and four pressures p_i, at the end-points of four equal Plateau borders of length L, as in figure 1. We write

$$\vec{q} = C\vec{p} \tag{5}$$

so flow conductance is represented by the 4×4 symmetric matrix C. Tetrahedral symmetry dictates that it has one non-degenerate eigenvalue λ_1 and a triply degenerate one λ_2. The first is associated with equal pressures at the end of each arm: clearly this gives zero flow. Only a single finite eigenvalue remains.

Without loss of generality we can represent the elements of C as

$$C_{ij} = c\left(\delta_{ij} - \frac{1}{4}\right) \tag{6}$$

for which $\lambda_1 = 0$ and $\lambda_2 = c$.

Note that when the p_i are all equal, each $q_i = 0$ as required. If $p_1 = -p_2$, $p_3 = p_4 = 0$, we have a situation where liquid flows into one arm and out of another, with no flow in the remaining two arms. In this case (6) gives $q_1 = -q_2 = p_1 c, q_3 = q_4 = 0$. For the calculations described below we shall use $p_2 = p_3 = p_4 = -p_1/3$ which has $q_2 = q_3 = q_4 = -q_1/3 = -cp_1/3$; that is, flow into a single arm and equal flow out of the other three.

In each case the constant c characterizes the conductance properties of the vertex. We define the resistance parameter R of the vertex to be $1/c$. But how does c depend upon L?

As the length L of the arms is taken to infinity, the resistance R tends asymptotically to a linear form, because a resistance proportional to the change in L is being added to each arm, consistent with (4). This can be made particularly obvious by appeal to the second of the cases discussed above, for flow into a single arm and out of only one other. Hence

$$R_{pb} = \frac{\mathrm{d}}{\mathrm{d}L}\left(\frac{1}{c}\right). \tag{7}$$

A plot of $1/c$ against L will therefore tend asymptotically to a straight line; however, this line will not pass through the origin. Instead, its intercept with the L axis is the effective length correction. That is,

$$R = R_{\mathrm{vertex}} + LR_{pb} \tag{8}$$

or

$$R = R_{pb}(L + \Delta L). \tag{9}$$

The correction ΔL has the dimensions of length, but cannot depend upon L and is proportional to δ. We therefore non-dimensionalize with δ and plot

$$\frac{R}{R_{pb}\delta} = \frac{\Delta p}{Q}\frac{A_{pb}^2}{\eta\chi\delta} \tag{10}$$

in terms of L/δ. Once the length correction ΔL is established by computation, the length correction constant $\Delta L/\delta$ can be used in a theory which otherwise ignores the vertex.

2.2. *Free-boundary case*

We shall solve the free-boundary case in a similar way here, but, since the line resistance is zero in this case, we must express the results in a form other than that of a length correction. We write the solution as a resistance parameter R for the vertex, which is independent of L. The constant R characterizes the flow through the vertex.

Dimensionally (essentially as argued in [1]), the flow conductance is of order δ^3, so R is of order δ^{-3} in this case. Whereas in the Poiseuille case we found a correction to a known resistance, here it stands on its own.

Therefore we write the resistance parameter as

$$R = r_v\eta\delta^{-3} \tag{11}$$

where the constant r_v will be found from the numerical calculations:

$$r_v = \frac{\Delta p\,\delta^3}{Q\eta}. \tag{12}$$

3. Computation of length corrections

The *Surface Evolver* package [21] was used to generate the tetrahedral vertex for the computations, with $\delta = 2 \times 10^{-4}$ m. After three levels of refinement the junction is approximated by a tessellation of 3584 facets. It is adapted [22] to a format recognized by *Gambit*, the mesh-creating part of the *Fluent* software. Difficulties arise when *Gambit* is used to import a further level of *Evolver* refinement. The mesh is finally imported into *Fluent*, where the boundary conditions are set. We treat the fluid in the junction as water.

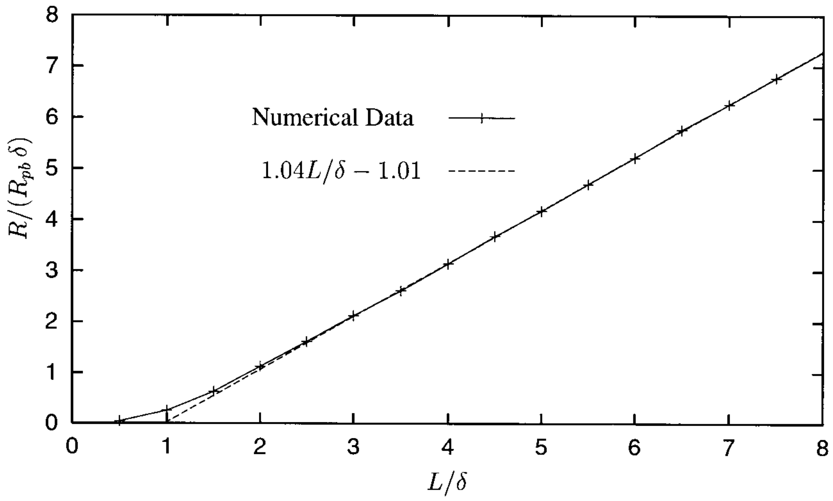

Figure 3. In the Poiseuille case, the straight-line fit through the numerical data for the flow resistance gives a length correction of $\Delta L/\delta \simeq 1.01/1.04 \simeq 0.97$.

3.1. *Poiseuille flow*

The computations used the configuration of figure 2 with $p_1 = 37.5$ kg m^{-1} s^{-2}. The flow rate was evaluated from the numerical solution: $Q = 1.70 \times 10^{-11}$ m^3 s^{-1}. The scaled flow resistance (10) is plotted against arm length L in figure 3, the length being represented by the scaled form L/δ.

The asymptotic linear form, given by

$$\frac{R}{R_{pb}\delta} = 1.04\frac{L}{\delta} - 1.01 \tag{13}$$

is rapidly approached. (Note the slight numerical inaccuracy—the slope should be equal to one.) In the absence of the divergence at small L, when the junction becomes important, the graph shows that $R = 0$ when $L \approx 0.97\delta$. This corresponds to a length correction constant of

$$\frac{\Delta L}{\delta} \approx 0.97. \tag{14}$$

3.2. *Free-boundary case*

This is computed in a similar way; the difference arises because we must express the boundary conditions in terms of a fluid velocity at the inlet. Due to convergence problems, our numerical solutions have been restricted to inflow velocities of the order of 1 m s^{-1}, giving a Reynolds number of the order of 10^2 which is hardly in the creeping flow regime upon which standard foam drainage theory is based. Nevertheless, we believe the value found for r_v is reliable, since little variation was found for a range of boundary conditions.

We see, as expected, flow of constant velocity v in the upper arm of the tetrahedral vertex, and flow with velocity $v/3$ in each of the lower arms. The flow rate is calculated and the pressure in each of the upper and lower arms found. (This latter quantity tends asymptotically to a constant in each arm, a short distance from the vertex.) The constant r_v is found from (12), for a range of pressure differences and corresponding flow rates, to be $r_v \approx 250 \pm 25$. The uncertainty is due to numerical error in the calculations.

4. Discussion

The incorporation of corrections based on the calculated constants, and comparison with experiment, should be straightforward. However, this requires careful consideration of the parameters defining the experimental conditions (e.g. viscosity). Preliminary studies suggest that in the Poiseuille case the effect of the length correction is, as expected, to slightly reduce the power-law exponent from the value of 0.5. In the case of free-boundary flow, it should be possible to supply the prefactor necessary to make a fully quantitative comparison of drainage theory with the experimental data of Koehler *et al* [1, 14]. Such comparisons will be presented in a further paper.

5. Conclusions

The evaluation of the constants necessary to make a first approximation to vertex effects moves the drainage theory a step forward. However, caution has always been advised in doing so. There remain yet more effects to be considered, including the distortion of the structure, as the foam becomes wetter. Furthermore, there is as yet little understanding of the underlying physics and chemistry that dictates the two limiting behaviours considered here.

Acknowledgments

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